The Importance of Cultivating Mathematical Thinking in Teaching

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ABSTRACT
Mathematics is very important at any stage of learning, as it not only enhances people's computational abilities but also improves their logical thinking abilities. The difference between mathematics learning in university and other stages is not only in terms of content, but also in terms of learning methods and levels. In order to learn mathematics well during university, it is particularly important for the students to cultivate mathematical thinking in addition to mastering certain learning methods and skills. And good cultivation of mathematical thinking can stimulate students' creativity and better use mathematical tools to solve practical problems.

Keywords: Mathematical thinking, Circuitous thinking, Expansive thinking.

1. INTRODUCTION

From elementary school to university, mathematics is ubiquitous in people’s lives. Whether it is shopping, traveling, or exercising, mathematics cannot be separated from its application. Therefore, mathematics learning has always been continuous. In the process of learning mathematics, it is necessary to continuously learn various basic methods and formulas, strengthen consolidation and understanding, in order to apply what has been learned. Mathematics learning is not static at different stages, as the requirements for knowledge mastery vary. It is necessary to adjust different learning methods for different stages. The learning mode before university is exam-oriented education, which is a teacher-led teaching model, while the learning during university is a student-centered combination of learning and teaching. After teaching key points in class, students need to absorb and integrate them on their own, rather than rote memorization. There are significant differences between individual students in the classroom, and there are also significant differences between several classes in the same classroom. Some classes are concentrated and fail, so it is particularly important to gradually cultivate one's learning thinking in learning. The teacher constructs a mathematical framework for students through the problems that arise during the teaching process, and through subtle means, imparts learning methods to students, guiding them to think independently, gradually cultivating their thinking in learning mathematics, and then forming a certain knowledge system through careful exploration by students themselves outside of class.

2. THE CULTIVATION OF ANALOGICAL THINKING

Most students believe that learning mathematics only requires memorizing formulas and conclusions. However, from past math exams such as the middle school entrance examination, college entrance examination, and graduate school entrance examination, it can be found that mechanical memorization and application of formulas cannot achieve good results. In addition to understanding the basic use of formulas through the teacher's explanations in class, students also need to constantly use their brains, draw inferences from one problem, and use divergent thinking to lead to multiple problems. Only in this way can they fully understand the formulas and solve any problems. Therefore, helping students cultivate mathematical thinking in the classroom is crucial, and drawing inferences from one problem is particularly important in the cultivation of mathematical thinking.

For example, in the study of integral upper limit functions, the textbook provides a relatively simple introduction to this function, but in subsequent in-depth learning, the scope of use of this function is particularly broad. Merely applying basic formulas is not enough, which causes students to have no way to start when solving problems. What is the reason for this? After all is said and done, when learning the upper limit function of integrals, students only know the surface on and don't understand the essence of the function. For the integral upper limit function $\int_a^x f(t)dt$, from a formal perspective, it is necessary to distinguish between the integral upper limit and the integral variable; Essentially, it is also necessary to distinguish when x is constant and when it is variable. Therefore, during teaching, teachers guide students to first analyze functions. Formally speaking, it is represented by integration. Therefore, when integrating, the x serves as the upper limit, which is a constant. The value after integration changes with the changes of the x. At this point, x is the variable. In this way, students have a thorough understanding of the integral upper limit function itself. In the subsequent operations, as long as they clarify whether it is a pre-integration operation or a post-integration operation, they can know what the x is. So what special properties does it possess? $\Phi'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$ for taking the derivative of the integral upper limit function is derived through the definition formula of the derivative. [1] The derivation formula of the integral upper limit function is given, making its application scope wider. At this point, teachers can guide students to think, for example, if the upper limit of integration is replaced by a composite function of a mathematical formula, can the formula still be directly applied? Obviously wrong, how to solve such a problem? By guiding students to think step by step and discussing the form of the integral upper limit function itself, it can be found that after integrating the integral upper limit function, it is a composite function about $\varphi(x)$. By using the differentiation method of composite functions, the derivative can be obtained as an $f(x)\varphi'(x)$.

The derivation of this idea deepens students' understanding of the upper limit function of integrals. In order to better cultivate their mathematical thinking, students should be guided to continue exploring this idea. When solving the problem, teachers should emphasize that only by fully conforming to the form of the formula can people apply known formulas and cannot blindly use them. For example, when judging the monotonicity of $F(x) = \int_0^x (x-t)f(t)dt$ for functions, obviously, many students see this question and believe that using the derivative sign of the integral upper limit function can directly determine monotonicity. But upon careful observation, it can be found that there are more x in this integrand function than in the formula, not just variable t. If people directly apply formulas, $F'(x) = 0$, this is an error. Why does this error occur? Students still haven't figured out the "identity" of x. Therefore, teachers will guide students back to discussing about what time x is a variable and what time it is a constant. It is obvious that $F'(x)$ are required before integration can work. So what about the x before integration? It is a constant because the x interferes with people when calculating the derivative, so they can adjust the x to the appropriate position before calculating the derivative. Before integration, it is a constant that can be adjusted outside the integration sign. This way, the integrand only contains t, and $F(x) = x\int_0^x f(t)dt - \int_0^x tf(t)dt$, and then when people take the derivative, they can use the known derivative formula. Therefore, for problems with integral upper limit functions included in known conditions or conclusions, the general direct approach is to first calculate the derivative of the integral upper limit function. If it does not conform to the form of the known formula, the x of the quantity unrelated to the integral variable should be first mentioned outside the integral sign, and then the derivative rule should be used to calculate the derivative. For integral upper limit functions that cannot be directly converted, variable substitution should be adopted to convert them into formula form and then calculate the derivative.

Through the above analysis, it can be found that following the analytical approach and starting from the basic formula, people can extend the integral upper limit function from the most basic and simple formula to various situations, which can solve a large number of practical problems. Furthermore, the derivative of the integral lower limit function and the variable upper and lower limit function have also been solved. From this, it can be seen that the study of mathematics cannot rely on rote memorization. It is necessary to carefully understand the essence of formulas, draw analogies, and apply them flexibly.

3. THE CULTIVATION OF CIRCUITOUS THINKING

The learning process of mathematics is not static. When this method is not feasible, people can consider changing their approach, even thinking in a roundabout way, which can open a new door to people's thinking. When some students first started learning the existence theorem of the real number system, they had not yet switched their thinking from studying high school mathematics and found it very abstract and difficult to understand. They even felt that such an abstract theorem had no meaning. At this point, teachers need to tell students to wait, not to worry, to put it aside for now. When teachers introduce limits and continuity, they can guide students to review this theorem again, and then they will discover its clever use. Adopting a circuitous way of thinking is very useful when encountering unfamiliar and difficult to understand mathematical definitions at the beginning. By combining the questions that confuse people at the beginning with the knowledge learned later, and through repeated contemplation and thinking, they will find that the previous knowledge is gradually understood.

For example, when learning to calculate the volume of a solid, the basic knowledge points that students are required to master during the teaching process are the $V = \pi\int_a^b f^2(x)dx$ for the volume of a solid rotating around a coordinate axis. When using this formula, it is required that the planar figure contains this number axis and rotates around it, so that the obtained volume can be directly obtained by applying the formula. However, in practical applications, it is difficult to obtain the inverse function of some functions, such as finding the volume of a plane shape enclosed by a sine function and the horizontal axis rotating around the vertical axis. It is obvious that the plane shape does not include the vertical axis. In this case, by constructing a large volume minus a small volume, the known formula can be directly applied. However, in practical operation, it is found that the anti-sine function must be given within a specific interval, which causes students some trouble in calculating. In addition, there are also difficulties in constructing projection intervals for unknown placeholders. For example, when solving $y = f(x)$, it is difficult to obtain the volume of a solid formed by rotating the plane shape enclosed by the horizontal axis around the vertical axis. It is obvious that the enclosed plane shape is difficult to obtain in the projection interval of the vertical axis, which makes it impossible to apply basic formulas. At this point, teachers need to tell the students to pause and adopt a different way of thinking. When the problem cannot be solved directly, people should consider from the side. Since it is difficult to obtain the projection interval on the vertical axis, we should project on the relatively easy number axis based on the shape of the graph. So, based on the shape of the graph, project on a relatively easy number axis and follow the idea of deriving the original formula. If any small area is selected within the projection interval, there will be a small rectangle corresponding to the small area. This small rectangle rotates around the horizontal axis to obtain a thin, hollow cylinder. It is unfolded to obtain a cylinder with a thickness of dx with an upper surface of $2\pi x f(x)$. Its approximate volume is $2\pi x f(x)dx$. By integrating in the projection interval, the volume of the rotating body is obtained.

Through circuitous thinking, it can be found that the problems that cannot be solved directly can be solved well through circuitous thinking from other directions. Therefore, cultivating students' mathematical thinking during the teaching process can deepen their understanding of knowledge from multiple aspects and achieve twice the result with half the effort.

4. THE CULTIVATION OF CONTRASTIVE AND EXPANSIVE THINKING

Mathematics is a subject with strong logic and is difficult to understand. During the learning process, one will encounter a large number of formulas, properties, theorems, etc. If one relies on mechanical memorization and blind application, it is difficult to master the essence of mathematics. It is necessary to pay attention to understanding and understanding the meaning of formulas when memorizing them, and also focus on the connection of knowledge before and after. For example, when memorizing the indefinite integral formula of trigonometric functions, rote memorization of a large number of formulas will only make our minds more confused. People can go through the derivation process in their hearts, so as to not only remember the formulas but also consolidate the methods. Therefore, paying attention to the similarities and differences of problems in learning and using them in a reasonable combination is the key to mastering mathematics. When encountering practical problems, the teaching process should actively guide students to clarify where and how to start, and cultivate their expanded thinking through comparison. The process of learning mathematics is an accumulation process. With the continuous acquisition of knowledge, the methods of solving the same problem will also change. People need to constantly summarize and generalize, expand their thinking through comparative methods, and reinforce their impressions.

For example, when judging the roots of an equation, when just touching the zero point theorem, based on the conclusion of the zero point theorem, it is easy to think of constructing an auxiliary function through the conclusion equation. This auxiliary function is the function discussed in the theorem. When the conditions of the zero point theorem are met, the situation of the equation roots can be judged. But with the introduction of Rolle's theorem, it will be found that the conclusion of Rolle's theorem is also given in the form of an equation, and the difference between them is that one is in the form of the original function and the other is in the form of the derivative function. If the corresponding rule of the function in the equation is given in symbolic form, people can easily distinguish which theorem to use for judgment. However, when the equation is represented in specific functional form, it is difficult to distinguish whether it is the form after differentiation or the form before differentiation. Therefore, both theorems can solve the problem of equation roots. If determining that $e^x - 2 = x$ of an equation has at least one root in $(0,2)$, it is difficult to distinguish whether it is a derivative equation or an undifferentiated equation. If it is considered an undifferentiated equation, then the auxiliary function constructed using the zero point theorem is $f(x) = e^x - 2 - x$. If it is considered as an equation after differentiation, then use Rolle's theorem to construct an auxiliary function as $f(x) = e^x - 2x - \frac{1}{2}x^2$.

It can be seen from this that as people accumulate knowledge, there will be many different methods to solve the same problem. There is a must to constantly reform and innovate, so that the knowledge can be unified and integrated with each other.

5. CONCLUSION

Studying mathematics during university is very important. Learning mathematics is not just about memorizing formulas and mastering calculations, but also requires teachers to teach in class, integrate and apply knowledge from one example to another, in order to learn by analogy. The cultivation of mathematical thinking during university not only lays a solid foundation for further mathematical research, but also exercises students' ability to constantly innovate. Through mastering mathematical knowledge in other subjects, it helps guide and inspire students. The way of thinking and research methods cultivated in mathematics learning will always affect students' learning and work. Therefore, in mathematics teaching, it is necessary to deeply integrate its methods into students' thinking, guide and help students summarize mathematical learning methods, cultivate good mathematical habits, and lay a solid foundation for better learning of deep level mathematics courses.

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